# Mobile Communications

## TCS 455


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**Lecture 24**

**Office Hours:**
**BKD 3601-7**
**Tuesday 14:00-16:00**
**Thursday 9:30-11:30**

# Announcements

- Read
  - Chapter 9: 9.1 – 9.5
  - Section 1.2 from [Bahai, *Multi-carrier Digital Communications: Theory And Applications Of OFDM*, 2002]
    - Uploaded to the SIIT online lecture note system.

# Chapter 5

## OFDM

**Office Hours:**
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**Tuesday 14:00-16:00**
**Thursday 9:30-11:30**

# OFDM

- Let $S_1$, $S_2$, …, $S_N$ be the information symbol.
- The discrete baseband OFDM modulated symbol can be expressed as

Some references may use different constant in the front

$$s(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left( j \frac{2\pi k t}{T_s} \right), \quad 0 \le t \le T_s$$

$$= \sum_{k=0}^{N-1} S_k \underbrace{\frac{1}{\sqrt{N}} 1_{[0,T_s]}(t) \exp\left( j \frac{2\pi k t}{T_s} \right)}_{c_k(t)}$$

Some references may start with different time interval, e.g. [-$T_s$/2, +$T_s$/2]

Note that:

$$\mathrm{Re}\{s(t)\} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left( \mathrm{Re}\{S_k\} \cos\left( \frac{2\pi k t}{T_s} \right) - \mathrm{Im}\{S_k\} \sin\left( \frac{2\pi k t}{T_s} \right) \right)$$

# Chapter 5
## OFDM

Wireless Channel

**Office Hours:**
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**Thursday 9:30-11:30**

# Three steps towards OFDM

1. Solve Multipath → Multicarrier modulation (FDM)
2. Gain Spectral Efficiency → Orthogonality of the carriers
3. Achieve Efficient Implementation → FFT and IFFT

# Chapter 5
## OFDM

Multi-Carrier Transmission

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# FDM: Better or Worse?

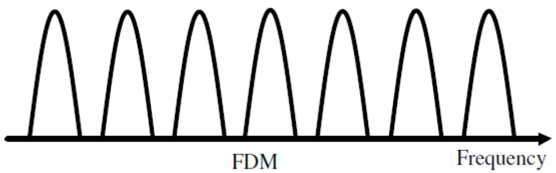
- Comparison with a single higher rate serial scheme
  - The parallel system, if built straightforwardly as several transmitters and receivers, will certainly be more costly to implement.
  - Each of the parallel subchannels can carry a low signalling rate, proportional to its bandwidth.
    - The sum of these signalling rates is less than can be carried by a single serial channel of that combined bandwidth because of the unused guard space between the parallel sub-carriers.
  - The single channel will be far more susceptible to inter-symbol interference.
    - This is because of the short duration of its signal elements and the higher distortion produced by its wider frequency band, as compared with the long duration signal elements and narrow bandwidth in sub-channels in the parallel system.

# FDM (3)

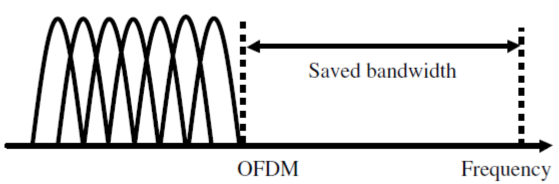
- Before the development of equalization, the parallel technique was the preferred means of achieving high rates over a dispersive channel, in spite of its high cost and relative bandwidth inefficiency.

# OFDM

- OFDM = Orthogonal frequency division multiplexing
- One of multi-carrier modulation (MCM) techniques
  - Parallel data transmission (of many sequential streams)
  - A broadband is divided into many narrow sub-channels
  - Frequency division multiplexing (FDM)




- High spectral efficiency
  - The sub-channels are made orthogonal to each other over the OFDM symbol duration $T_s$.
    - Spacing is carefully selected.
  - Allow the sub-channels to overlap in the frequency domain.
  - Allow sub-carriers to be spaced as close as theoretically possible.



# OFDM: Orthogonality

$$\int c_{k_1}(t)c_{k_2}^*(t)dt = \int_0^{T_s} \exp\left(j\frac{2\pi k_1 t}{T_s}\right)\exp\left(-j\frac{2\pi k_2 t}{T_s}\right)dt$$

$$= \int_0^{T_s} \exp\left(j\frac{2\pi(k_1-k_2)t}{T_s}\right)dt = \begin{cases} T_s, & k_1 = k_2 \\ 0, & k_1 \neq k_2 \end{cases}$$

When $k_1 = k_2$,

$$\int c_{k_1}(t)c_{k_2}^*(t)dt = \int_0^{T_s} 1dt = T_s$$

When $k_1 \neq k_2$,

$$\int c_{k_1}(t)c_{k_2}^*(t)dt = \frac{T_s}{j2\pi(k_1-k_2)}\exp\left(j\frac{2\pi(k_1-k_2)t}{T_s}\right)\Bigg|_0^{T_s}$$

$$= \frac{T_s}{j2\pi(k_1-k_2)}(1-1) = 0$$

# Frequency Spectrum

$$s(t) = \sum_{k=0}^{N-1} S_k \underbrace{\frac{1}{\sqrt{N}} 1_{[0,T_s]}(t) \exp\left( j\frac{2\pi k t}{T_s} \right)}_{c_k(t)}$$

$$\Delta f = \frac{1}{T_s}$$

**This is the term that makes the technique FDM.**

$$1_{\left[-\frac{T_s}{2},\frac{T_s}{2}\right]}(t) \xrightarrow{\mathcal{F}} T_s \operatorname{sin}c\left(\pi T_s f\right)$$

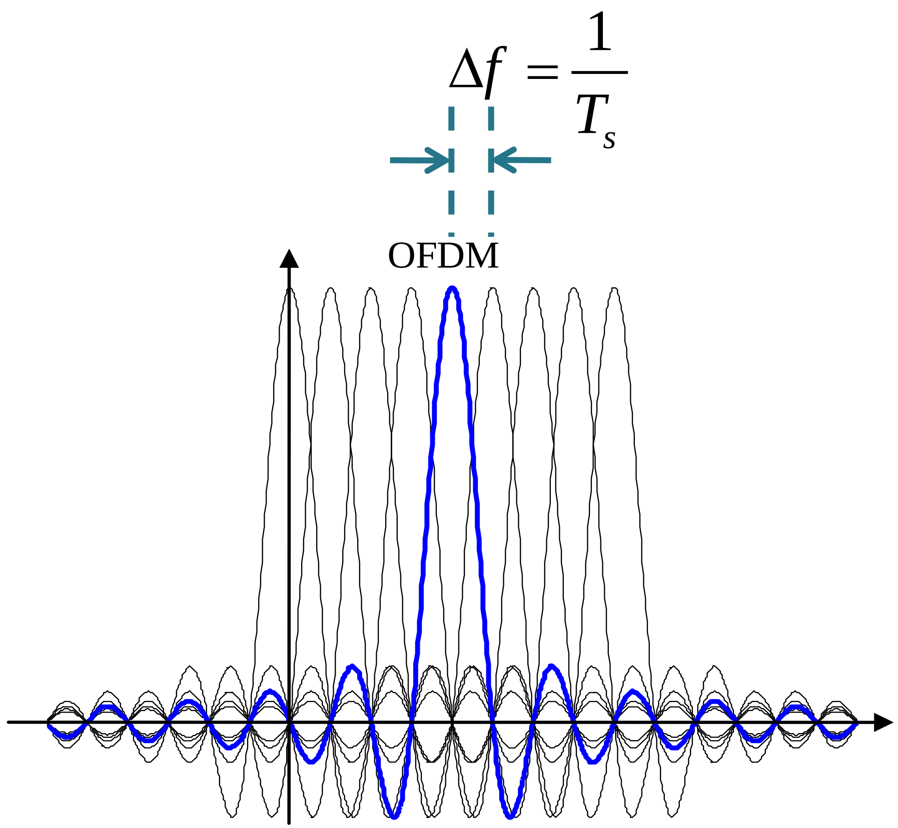
$$c(t) = \frac{1}{\sqrt{N}} 1_{[0,T_s]}(t) \xrightarrow{\mathcal{F}} C(f) = \frac{1}{\sqrt{N}} T_s e^{-j2\pi f\frac{T_s}{2}} \operatorname{sin}c\left(\pi T_s f\right)$$

$$c_k(t) = c(t)\exp\left( j\frac{2\pi k t}{T_s} \right) \xrightarrow{\mathcal{F}} C_k(f) = C\left( f - \frac{k}{T_s} \right) = C\left(f - k\Delta f\right)$$

$$s(t) = \sum_{k=0}^{N-1} S_k c_k(t) \xrightarrow{\mathcal{F}} S(f) = \sum_{k=0}^{N-1} S_k C_k(f)$$

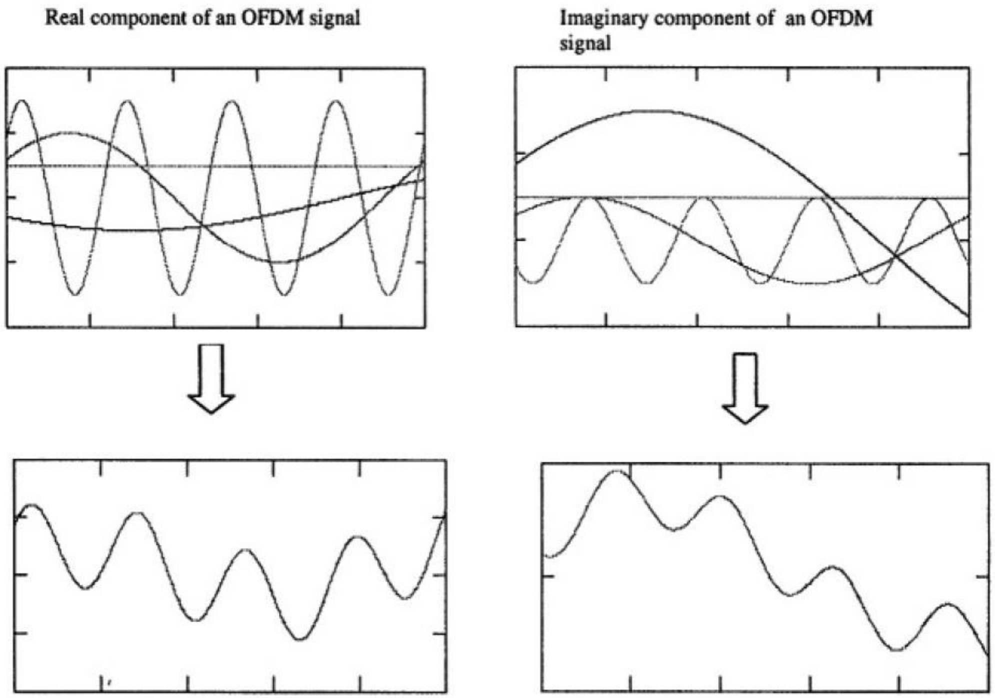
$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k e^{-j2\pi\left(f-k\Delta f\right)\frac{T_s}{2}} T_s \operatorname{sin}c\left(\pi T_s\left(f - k\Delta f\right)\right)$$

# Subcarrier Spacing




Spectrum Overlap in OFDM

# Time-Domain Signal




[Bahai, 2002, Fig 1.7]

Real and Imaginary components of an OFDM symbol is the *superposition of several harmonics* modulated by data symbols

$$s(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left( j\frac{2\pi kt}{T_s} \right), \quad 0 \le t \le T_s$$

$$\mathrm{Re}\{s(t)\} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left( \mathrm{Re}\{S_k\} \cos\left( \frac{2\pi kt}{T_s} \right) - \mathrm{Im}\{S_k\} \sin\left( \frac{2\pi kt}{T_s} \right) \right)$$

# Summary

- So, we have a scheme which achieve
  - Large symbol duration ($T_s$) and hence less multipath problem
  - Good spectral efficiency
- One more problem:
  - There are so many carriers!

# Discrete Fourier Transform (DFT)

$$s(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left( j \frac{2\pi kt}{T_s} \right), \quad 0 \le t \le T_s$$

Sample the signal in time domain:

$$s[n] = s\left( n \frac{T_s}{N} \right) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left( j \frac{2\pi k}{\cancel{T_s}} n \frac{\cancel{T_s}}{N} \right)$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left( j \frac{2\pi kn}{N} \right) = \sqrt{N}\ \mathrm{IDFT}\{S\}[n]$$

# DFT

## 5 Discrete Fourier transform (DFT)

In DFT, we work with $N$-point signal (finite-length sequence of length $N$) in both time and frequency domain. To simplify the definition we define

$$\boxed{\psi_N = e^{j\frac{2\pi}{N}}}$$

and the DFT matrix $Q = \Psi_N$ whose element on the $p$th row and $q$th column is given by $\psi_N^{-(p-1)(q-1)}$:

$$\boxed{\Psi_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \psi_N^{-1} & \psi_N^{-2} & \cdots & \psi_N^{-(N-1)} \\ 1 & \psi_N^{-2} & \psi_N^{-4} & \cdots & \psi_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \psi_N^{-(N-1)} & \psi_N^{-2(N-1)} & \cdots & \psi_N^{-(N-1)(N-1)} \end{bmatrix}}$$

### 5.2. Properties of $\Psi_N$

- $\boxed{\Psi_N^{-1} = \frac{1}{N}\Psi_N^*.}$ Equivalently, $\Psi_N^{-1}\Psi_N = NI_N$.

# DFT

**Definition 5.3.** The $N$-point DFT of an $N$-point signal (column vector) $x$ is given by

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-jnk\frac{2\pi}{N}} = \boxed{\sum_{n=0}^{N-1} x[n]\, \psi_N^{-nk}} \;; 0 \le k < N.$$

The inverse DFT is given by

$$\boxed{\underset{0\le n<N}{x[n]} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, \psi_N^{nk}} \underset{\mathrm{DFT}^{-1}}{\overset{\mathrm{DFT}}{\rightleftharpoons}} \underset{0\le k<N}{X[k]} = \sum_{n=0}^{N-1} x[n]\, \psi_N^{-nk}$$

In matrix form,

$$\boxed{x = \frac{1}{N}\Psi_N^{*} X \underset{\mathrm{DFT}^{-1}}{\overset{\mathrm{DFT}}{\rightleftharpoons}} X = \Psi_N \times x.}$$

# DFT

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In matrix form,

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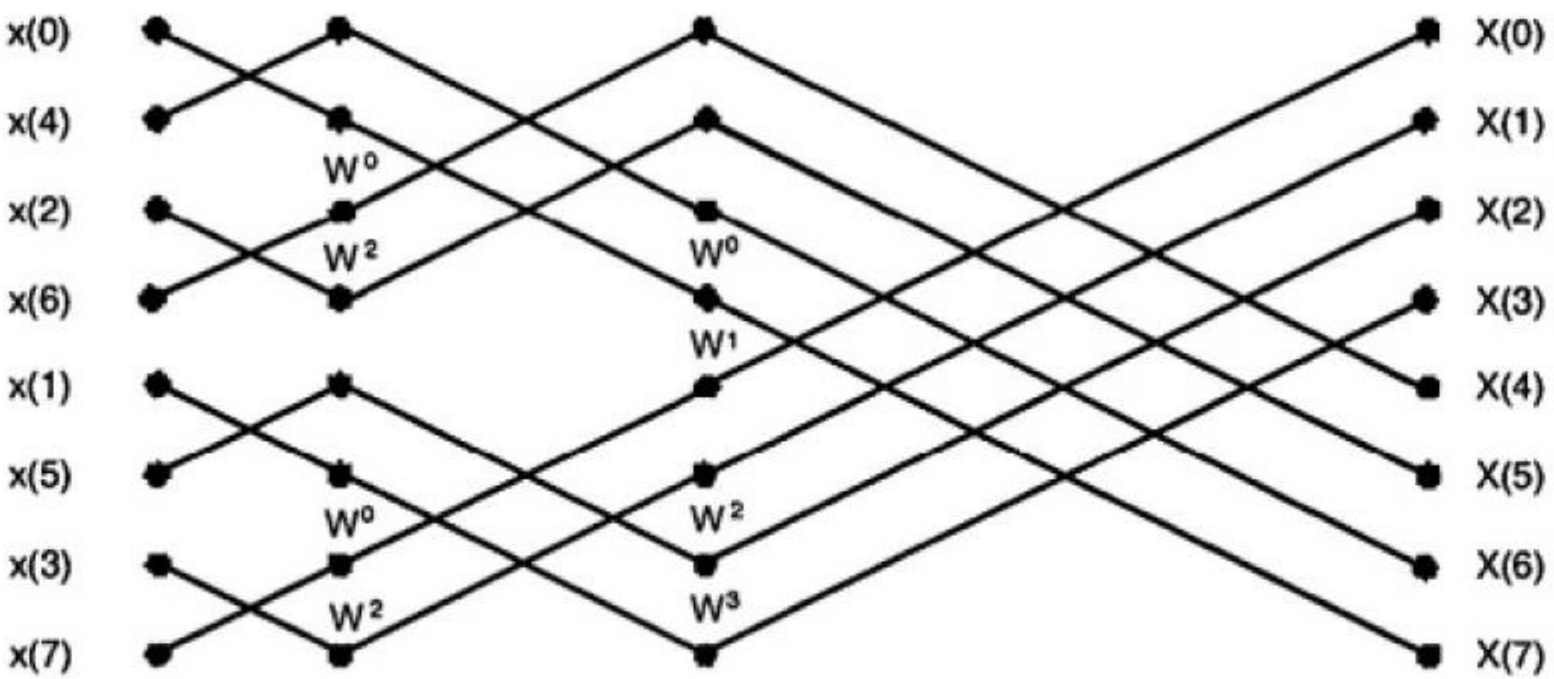
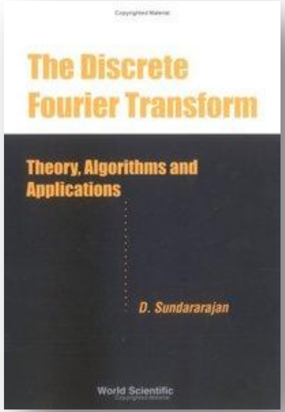
# Efficient Implementation: (I)FFT




[Bahai, 2002, Fig. 2.9]

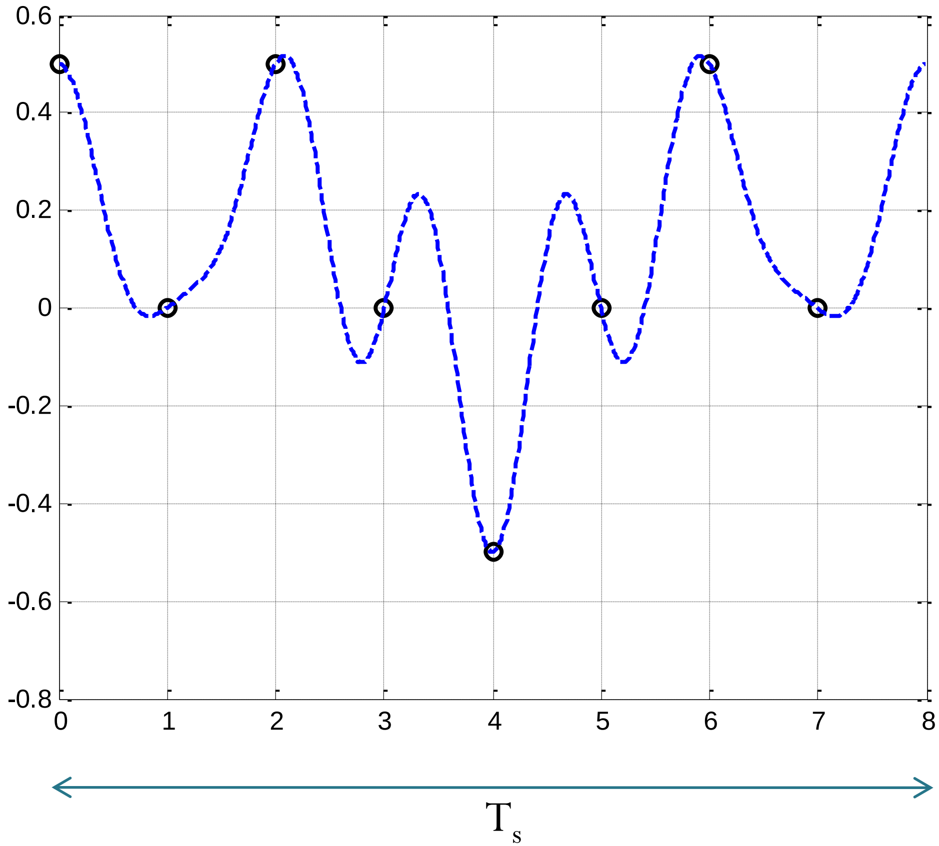
# FFT

- The history of the FFT is complicated.
- As with many discoveries and inventions, it arrived before the (computer) world was ready for it.
- Usually done with $N$ a power of two.
  - Not only is it very efficient in terms of computing time, but is ideally suited to the binary arithmetic of digital computers.
- References: E. Oran Brigham, *The Fast Fourier Transform*, Prentice-Hall, 1974.




# DFT Samples

- Here are the points s[n] on the continuous-time version s(t):



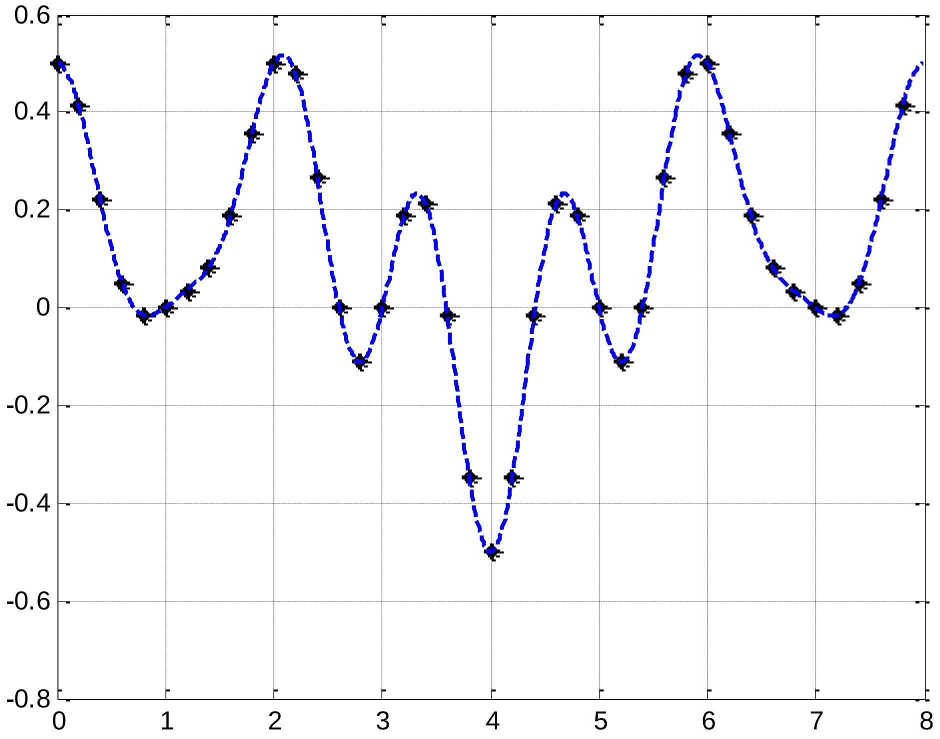

$$s[n] = s\left(n\frac{T_s}{N}\right)$$

$$= \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} S_k \exp\left(j\frac{2\pi kn}{N}\right)$$

$$= \sqrt{N}\,\mathrm{IDFT}\{S\}[n]$$

$$0 \le n < N$$

# Oversampling



# Oversampling (2)

- Increase the number of sample points from N to LN on the interval $[0, T_s]$.
- **L** is called the **over-sampling factor**.

$$s[n] = s\left(n\frac{T_s}{N}\right) \qquad 0 \le n < N$$

$$\Rightarrow$$

$$s^{(L)}[n] = s\left(n\frac{T_s}{LN}\right) \qquad 0 \le n < LN$$

$$s^{(L)}[n] = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} S_k \exp\left(j\frac{2\pi k}{\cancel{T_s}} n \frac{\cancel{T_s}}{LN}\right) = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} S_k \exp\left(j\frac{2\pi kn}{LN}\right)$$

$$= \frac{1}{\sqrt{N}} LN\left(\frac{1}{LN}\sum_{k=0}^{N-1} S_k \exp\left(j\frac{2\pi kn}{LN}\right)\right)$$

$$= L\sqrt{N}\left(\frac{1}{LN}\left(\sum_{k=0}^{N-1} S_k \exp\left(j\frac{2\pi kn}{LN}\right) + \sum_{k=N}^{NL-1} 0\exp\left(j\frac{2\pi kn}{LN}\right)\right)\right)$$

$$= L\sqrt{N}\left(\frac{1}{LN}\sum_{k=0}^{NL-1} \tilde{S}_k \exp\left(j\frac{2\pi kn}{LN}\right)\right) = L\sqrt{N}\ \mathrm{IDFT}\left\{\tilde{S}\right\}[n]$$

Zero padding:

$$\tilde{S}_k = \begin{cases} S_k, & 0 \le k < N \\ 0, & N \le k < LN \end{cases}$$

# Oversampling: Summary

$N$ points

$$s[n] = s\left(n\frac{T_s}{N}\right) = \sqrt{N}\ \mathrm{IDFT}\{S\}[n]$$

$0 \le n < N$

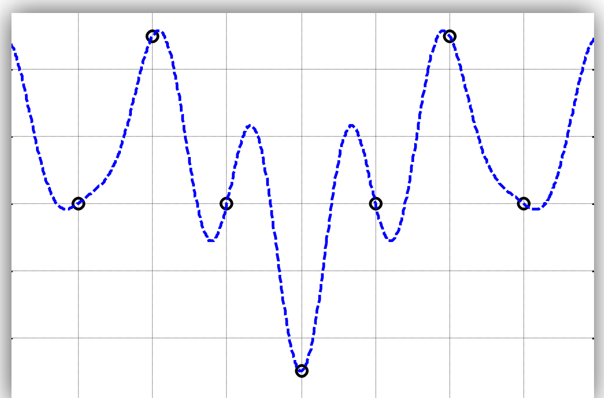
$LN$ points

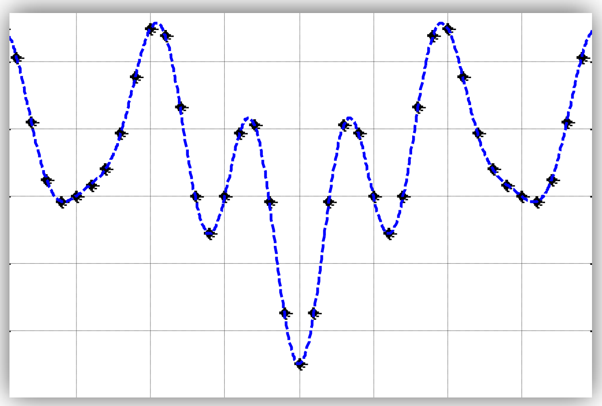
$$s^{(L)}[n] = s\left(n\frac{T_s}{LN}\right) = L\sqrt{N}\ \mathrm{IDFT}\{\tilde{S}\}[n]$$

$0 \le n < LN$

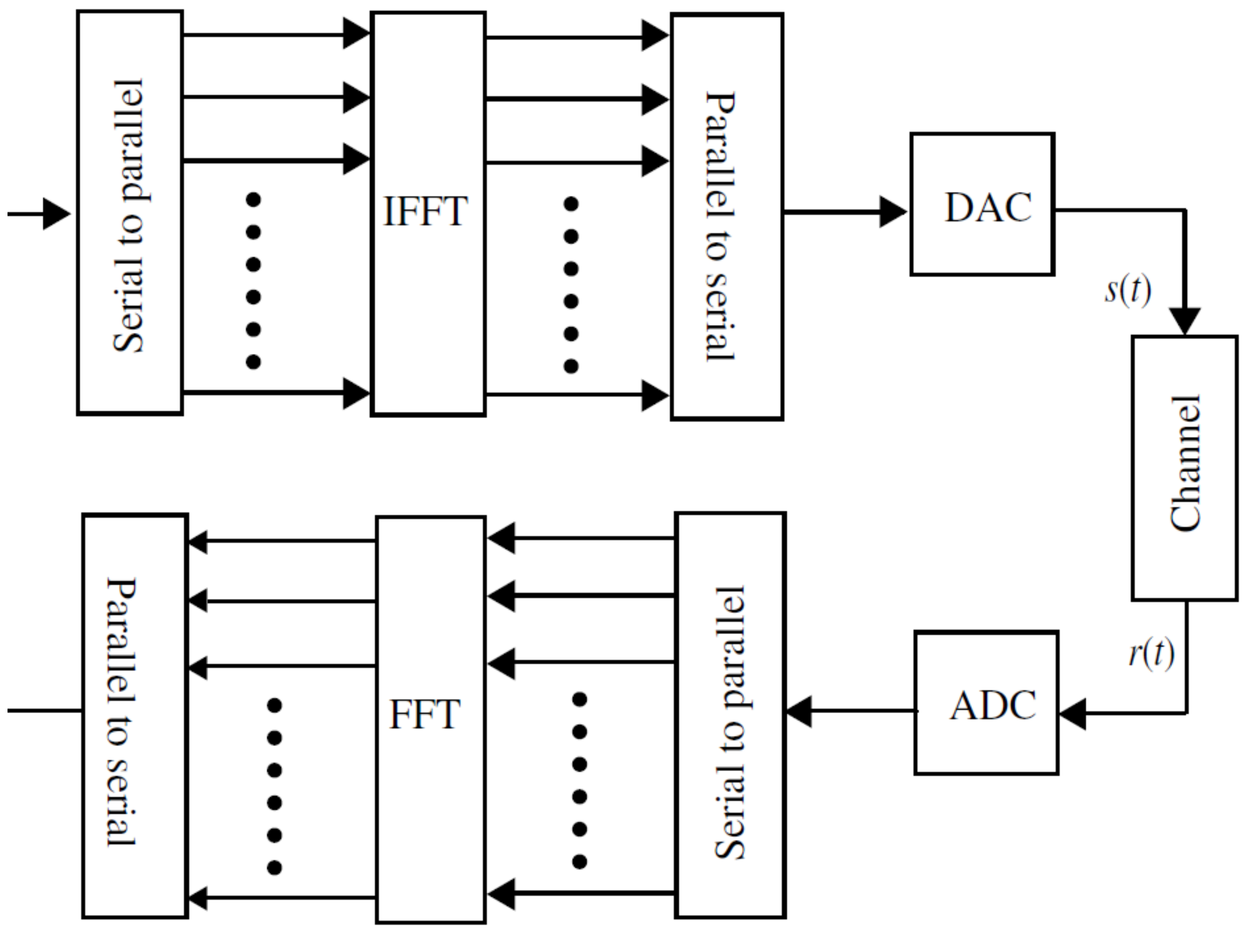
Zero padding:

$$\tilde{S}_k = \begin{cases} S_k, & 0 \le k < N \\ 0, & N \le k < LN \end{cases}$$





# OFDM implementation by IFFT/FFT



# Chapter 5
## OFDM

Cyclic Prefix

**Office Hours:**
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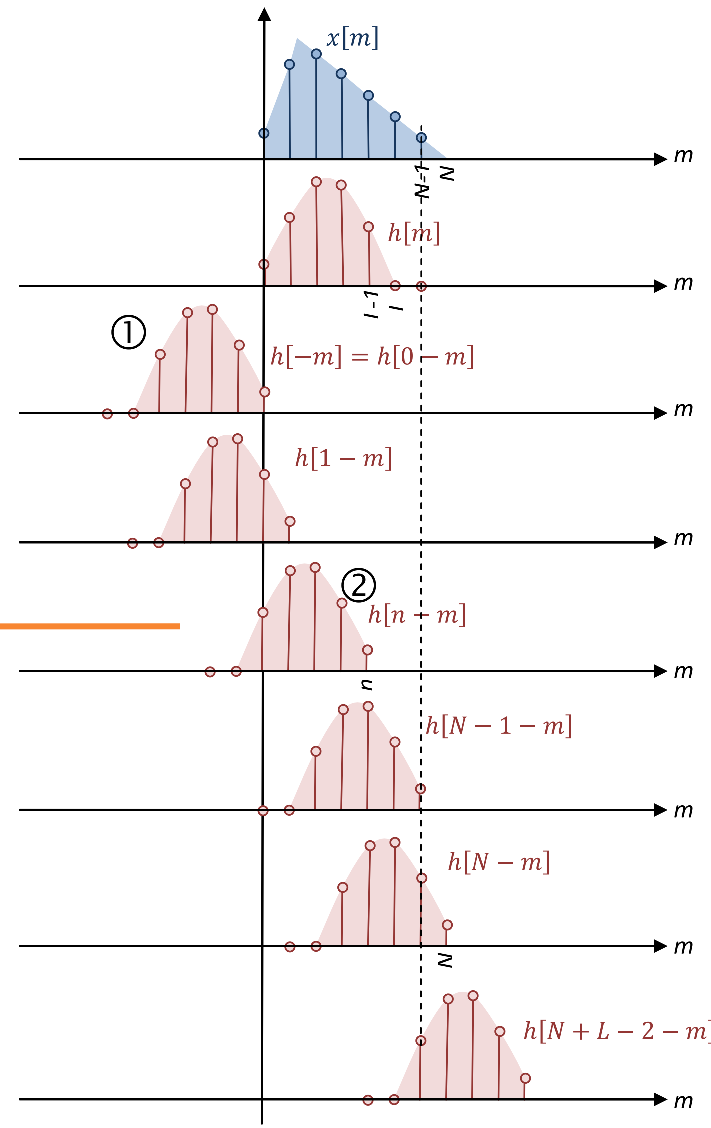
# Cyclic Prefix

- Can we "eliminate" the multipath problem?

# Convolution

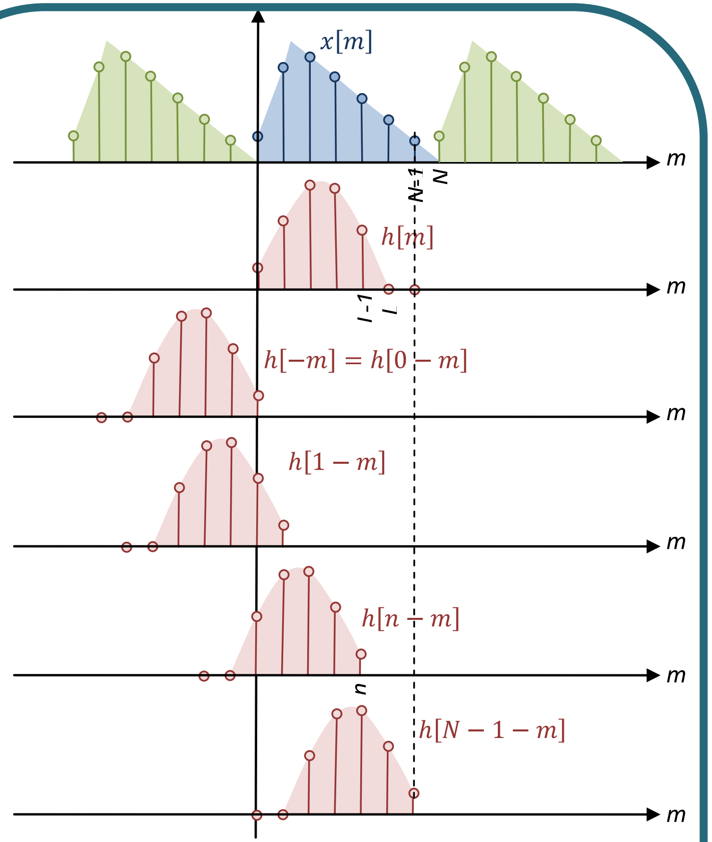
① Flip
② Shift
③ Multiply
④ Add




④ $\Sigma$ ③ $\otimes$

$$\{x * h\}[n] = \sum_m x[m]h[n-m]$$

# Circular Convolution




Replicate $x$
Then, perform the usual convolution
only on n = 0 to $N$-1